Topological Black Holes and Momentum Four Vector

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Abstract. We consider the energy-momentum definition of the Møller in both general relativity and teleparallel gravity to evaluate the energy distribution (due to both matter and fields including gravitation) associated with the topological black holes with a conformally coupled scalar field. Our results show that the energy depends on the mass M and charge Q of the black holes and cosmological constant Λ. In the some special limits, the expression of the energy reduces to the energy of the well-known space-times. The results also support the viewpoint of Lessner that the Møller energy-momentum formulation is a powerful concept of the energy-momentum. Furthermore, the energy obtained in teleparallel gravity is also independent of the teleparallel dimensionless coupling constants which means that it is valid not only in the teleparallel equivalent of the general relativity but also in any teleparallel model.

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1. Introduction

One of the most fundamental conserved quantities in physics is the energy-momentum which is associated with a symmetry of the space-time geometry. Moreover, the definition of an energy-momentum density for a gravitational field is one of the oldest and most controversial problems of gravitation. In both general relativity and teleparallel gravity, the definition of the energy-momentum has been associated with some debate because of the absence of a unique way of defining it.

Misner, Thorne and Wheeler [1] claimed that the energy is localizable only for spherical systems. But, Cooperstock and Sarracino [2] argued that if the energy is localizable in spherical systems, it is localizable in all systems. Furthermore, H. Bondi [3] has argued

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that general relativity does not permit a non- localizable form of energy, so in principle, an acceptable definition of the local energy-momentum density should exist. However, there is no generally accepted definition. A large number of formulations for the gravitational energy-momentum in both general relativity and teleparallel gravity has been given. Einstein [4], in 1916, gave the first energy-momentum complex used for evaluating energy and momentum in general relativity. Following his definition, several energy-momentum prescriptions have been proposed [5, 6, 7, 8, 9, 10, 11, 12]. There are doubts that these prescriptions could give acceptable results for a given space-time because of that most of the energy-momentum complexes are restricted to the use of particular coordinates. Therefore, the calculations must be carried out in Cartesian coordinates. Recently, the subject of the energy-momentum definition has been re-opened by Virbhadra and his colleagues [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. They have shown that several energy momentum complexes give the same and acceptable results for a given space-time. Virbhadra showed that Einstein, Landau and Lifshitz, Papapetrou, and Weinberg definitions give the same energy distribution as in the Penrose energy-momentum complex for a general non-static spherical symmetric metric of the Kerr-Schild class.

The Møller energy-momentum prescription does not necessitate carrying out calculation in "Cartesian" coordinates. Therefore, we can evaluate the energy distribution in any coordinate system. Lessner [31] claimed that the Møller complex is a powerful concept of the energy-momentum in general relativity. Teleparallel version of this formulation was obtained by Mikhail *et al* [32]. In his recent paper, Vargas [33] using the Einstein and Landau-Lifshitz complexes, calculated the energy-momentum density of the Friedman-Robertson-Walker space-time. Recently, Saltı, Aydogdu and their collaborators [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] have calculated energy-momentum density using different complexes for a given space-time in the teleparallel gravity.

The aim of this paper is to determine the energy of the universe on basis of the topological black holes with a conformally coupled scalar field, using the energy-momentum complex of Møller in Einstein's theory of general relativity and the teleparallel theory of gravity. In the next section, we introduce the topological black holes and carry out some necessary calculations for this model. In the section 3, we give the Møller energy-momentum definition in both general relativity and teleparallel gravity and then compute the energy-momentum density. Next, in the section 4, we give some special cases of the topological black holes. The last section is devoted to discussion and conclusion. Throughout this paper, Latin indices $(i, j, k,...)$ denote the vector number and Greek indices $(\mu, \nu, \lambda,...)$ represent the vector components. We use units in which $G = 1$, and $c = 1$.

2. Topological black holes

In four-dimensional asymptotically flat space-times, only horizons with the topology of a sphere are compatible with a well defined causal structure for black holes. This topological censorship can be, however, circumvented introducing a negative cosmological constant and a number of black holes with flat or hyperbolic horizons have been reported in four and higher dimensions [48, 49, 50, 51, 52, 53]. Moreover, this class of black holes is found in gravity theories containing higher powers of the curvature [54, 55, 56, 57]. The topology of the event horizon affects the black hole thermodynamics drastically. Thus, it could be an interesting exercise to evaluate the energy-momentum of topological black holes.

The Einstein-Maxwell system in four dimensions with a cosmological constant and a real conformally coupled self-interacting scalar field is described by the action

$$I[g_{\mu\nu},\phi,A_\mu]=\int d^4x\sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G}-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\frac{1}{12}R\phi^2-\alpha\phi^4-\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}\right] \quad (1)$$

where G and α are the Newton's constant and arbitrary coupling constant, respectively. The corresponding field equations are

$$G_{\xi\beta}+\Lambda g_{\xi\beta}=8\pi G(T^{\phi}_{\xi\beta}+T^{em}_{\xi\beta}) \quad (2)$$

$$g^{\xi\beta}\nabla_\xi\nabla_\beta\phi=\frac{1}{6}\phi R+4\alpha\phi^3 \quad (3)$$

$$\partial_\beta(\sqrt{-g}F^{\xi\beta})=0 \quad (4)$$

The field eqns. (2), (3) and (4) admit an exact static solution whose metric is given as [58]

$$ds^2=\left[-\frac{\Lambda r^2}{3}+\chi\left(1+\frac{G\mu}{r}\right)^2\right]dt^2-\left[-\frac{\Lambda r^2}{3}+\chi\left(1+\frac{G\mu}{r}\right)^2\right]^{-1}dr^2-r^2d\Omega^2 \quad (5)$$

with $-\infty<t<\infty$ and $r>0$. Here $d\Omega^2$ stands for the line element of the two-dimensional base manifold Σ which is assumed to be compact, without boundary, and of constant curvature χ that can be normalized to $\pm1,0$. This means that the surface Σ is locally isometric to the sphere S^2, flat space $\Re^2$, or to the hyperbolic manifold H^2 for $\chi=+1,0,-1$, respectively.
The integration constants q and μ are not independent since they must satisfy

$$q^2=\chi G\mu^2\left(1+\frac{2\pi\Lambda G}{9\alpha}\right) \quad (6)$$

and they are related to the electric charge Q and the mass M, respectively

$$Q=\frac{\varsigma}{4\pi}q \qquad \textit{and} \qquad M=-\chi\frac{\varsigma}{4\pi}\mu \quad (7)$$

where ς is the area of the base manifold Σ. The non-vanishing components of the Einstein tensor $G_{\mu\nu}(\equiv 8\pi T_{\mu\nu}$, where $T_{\mu\nu}$ is the energy-momentum tensor for the matter field described by a perfect fluid with density ρ, pressure p) are

$$G_{11}=\frac{r^2\Lambda+1-\chi+\frac{G\mu^2\chi}{r^2}}{\frac{\Lambda}{3}r^4-(G\mu+r)^2\chi} \quad (8)$$

$$G_{22} = -r^2\Lambda + \frac{G\mu^2\chi}{r^2} \tag{9}$$

$$G_{33} = \frac{(-r^4\Lambda + G\mu^2\chi)\sin^2\theta}{r^2} \tag{10}$$

$$G_{00} = -\frac{(r^4\Lambda - r^2(\chi-1) + G\mu^2\chi)(r^4\Lambda - 3(G\mu+r)^2\chi)}{3r^6} \tag{11}$$

3. Energy of the Topological black holes

The metric tensor $g_{\mu\nu}$ for the line-element (5) is given as

$$\begin{pmatrix} \left[-\frac{\Lambda r^2}{3} + \chi\left(1+\frac{G\mu}{r}\right)^2\right] & 0 & 0 & 0 \\ 0 & -\left[-\frac{\Lambda r^2}{3} + \chi\left(1+\frac{G\mu}{r}\right)^2\right]^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} \tag{12}$$

and its inverse $g^{\mu\nu}$ is

$$\begin{pmatrix} \left[-\frac{\Lambda r^2}{3} + \chi\left(1+\frac{G\mu}{r}\right)^2\right]^{-1} & 0 & 0 & 0 \\ 0 & -\left[-\frac{\Lambda r^2}{3} + \chi\left(1+\frac{G\mu}{r}\right)^2\right] & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta}. \end{pmatrix} \tag{13}$$

The general form of the tetrad, e_i^μ, having spherical symmetry, was given by Robertson [59]. In the Cartesian form it can be written as

$$e_0^0 = i\Upsilon, \qquad e_a^0 = \kappa x^a, \qquad e_0^\alpha = i\Pi x^\alpha, \qquad e_a^\alpha = \zeta\delta_a^\alpha + \Psi x^a x^\alpha + \epsilon_{a\alpha\beta}\Delta x^\beta \tag{14}$$

where $\Upsilon, \zeta, \kappa, \Pi, \Psi$, and Δ are functions of t and $r = \sqrt{x^a x^a}$, and the zeroth vector e_0^μ has the factor $i = \sqrt{-1}$ to preserve Lorentz signature. We impose the boundary condition that in the case of $r \to \infty$ the tetrad given above approaches the tetrad of Minkowski space-time, $e_a^\mu = \mathtt{diag}(i, \delta_a^\mu)$ (where $a = 1, 2, 3$). In the spherical, static and isotropic coordinate system $\mathbf{X}^1 = r\sin\theta\cos\phi$, $\mathbf{X}^2 = r\sin\theta\sin\phi$, $\mathbf{X}^3 = r\cos\theta$, the tetrad components of the topological black holes can be obtained from the line-element given in the Eq. (5), using the general coordinate transformation $e_{a\mu} = \frac{\partial \mathbf{X}^\nu}{\partial \mathbf{X}^\mu} e_{a\nu}$.

$$\begin{pmatrix} \frac{i}{\left(-\frac{\Lambda r^2}{3}+\chi\left(1+\frac{G\mu}{r}\right)^2\right)^{1/2}} & 0 & 0 & 0 \\ 0 & \frac{i}{\left(-\frac{\Lambda r^2}{3}+\chi\left(1+\frac{G\mu}{r}\right)^2\right)^{-1/2}} s\theta c\phi & \frac{1}{r}\ c\theta c\phi & -\frac{s\phi}{r s\theta} \\ 0 & \frac{i}{\left(-\frac{\Lambda r^2}{3}+\chi\left(1+\frac{G\mu}{r}\right)^2\right)^{-1/2}} s\theta s\phi & \frac{1}{r}\ c\theta s\phi & \frac{c\phi}{r s\theta} \\ 0 & \frac{i}{\left(-\frac{\Lambda r^2}{3}+\chi\left(1+\frac{G\mu}{r}\right)^2\right)^{-1/2}} c\theta & -\frac{1}{r}\ s\theta & 0 \end{pmatrix} \tag{15}$$

where $i^2 = -1$. Here, we have introduced following notation: $s\theta = sin\theta, c\theta = cos\theta, s\phi = sin\phi\ and\ c\phi = cos\phi$.

3.1. The Møller Energy in General Relativity

In general relativity, the energy-momentum complex of Møller [9, 10] is given by

$$M_{\beta}^{\zeta} = \frac{1}{8\pi}\Sigma_{\beta,\alpha}^{\zeta\alpha} \tag{16}$$

satisfying the local conservation laws

$$\frac{\partial M_{\beta}^{\zeta}}{\partial x^{\zeta}} = 0 \tag{17}$$

where the antisymmetric super-potential $\Sigma_{\beta}^{\zeta\alpha}$ is

$$\Sigma_{\beta}^{\zeta\alpha} = \sqrt{-g}[g_{\beta\lambda,\gamma} - g_{\beta\gamma,\lambda}]g^{\zeta\gamma}g^{\alpha\lambda}. \tag{18}$$

The locally conserved energy-momentum complex M_{β}^{ζ} contains contributions from the matter, non-gravitational and gravitational fields. M_0^0 is the energy density and M_a^0 are the momentum density components. The momentum four-vector of Møller is given by

$$P_{\beta} = \int\int\int M_{\beta}^{0} dxdydz. \tag{19}$$

Using Gauss's theorem, this definition transforms into

$$P_{\beta} = \frac{1}{8\pi}\int\int \Sigma_{\beta}^{0a}\mu_a dS. \tag{20}$$

where μ_a (where $a = 1, 2, 3$) is the outward unit normal vector over the infinitesimal surface element dS. P_i give momentum components P_1, P_2, P_3 and P_0 gives the energy.

Using the matrices given Eqs. (12) and (13) with Eq. (18) , the required non-vanishing component of the antisymmetric super-potential is

$$\Sigma_0^{01} = 2r\sin\theta\left[-\frac{\Lambda}{3}r^2 - \chi\left(1 + \frac{G\mu}{r}\right)\frac{G\mu}{r}\right]. \tag{21}$$

Taking into account the above result, one can easily find the energy and momentum components of the topological black holes in general relativity as following, respectively

$$E_M(r) = r\left[-\frac{\Lambda}{3}r^2 - \chi\left(1 + \frac{G\mu}{r}\right)\frac{G\mu}{r}\right]. \tag{22}$$

and

$$M_1^0 = M_2^0 = M_3^0 = 0 \tag{23}$$

3.2. The Møller Energy in Teleparallel Gravity

The teleparallel theory of gravity is an alternative approach to gravitation and corresponds to a gauge theory for the translation group based on Weitzenböck geometry [60]. In the theory of teleparallel gravity, gravitation is attributed to torsion [61], which plays the role of a force [62], and the curvature tensor vanishes identically. The essential field is acted by a nontrivial tetrad field, which gives rise to the metric as a by-product. The translational gauge potentials appear as the nontrivial item of the tetrad field, so induces on space-time a teleparallel structure which is directly related to the presence of the gravitational field. The interesting place of teleparallel theory is that, due to its gauge structure, it can reveal a more appropriate approach to consider some specific problem. This is the situation, for example, in the energy and momentum problem, which becomes more transparent.

Møller modified general relativity by constructing a new field theory in teleparallel space. The aim of this theory was to overcome the problem of the energy-momentum complex that appears in Riemannian Space [63, 64]. The field equations in this new theory were derived from a Lagrangian which is not invariant under local tetrad rotation. Saez [65] generalized Møller theory into a scalar tetrad theory of gravitation. Meyer [66] showed that Møller theory is a special case of Poincare gauge theory [67, 68, 69].

In the teleparallel gravity, Møller's super-potential is given by Mikhail *et al.* [32] as

$$U_{\zeta}^{\eta\beta} = \frac{(-g)^{1/2}}{2\kappa} P_{\lambda\alpha\sigma}^{\tau\eta\beta}[\Psi^{\alpha} g^{\sigma\lambda} g_{\zeta\tau} - \Upsilon g_{\tau\zeta}\xi^{\lambda\alpha\sigma} - (1-2\Upsilon) g_{\tau\zeta}\xi^{\sigma\alpha\lambda}] \tag{24}$$

where $\xi_{\alpha\beta\lambda} = e_{i\alpha} e^{i}{}_{\beta;\lambda}$ is the con-torsion tensor and $e_i{}^{\mu}$ is the tetrad field and defined uniquely by $g^{\alpha\beta} = e_i^{\alpha} e_j^{\beta} \eta^{ij}$ (here η^{ij} is the Minkowski space-time). κ is the Einstein constant and Υ is free-dimensionless coupling parameter of teleparallel gravity. For the teleparallel equivalent of general relativity, there is a specific choice of this constant.

Ψ_{μ} is the basic vector field given by

$$\Psi_{\mu} = \xi^{\rho}{}_{\mu\rho} \tag{25}$$

and $P_{\lambda\eta\sigma}^{\tau\nu\beta}$ can be defined as

$$P_{\lambda\eta\sigma}^{\tau\nu\beta} = \delta_{\lambda}^{\tau} g_{\eta\sigma}^{\nu\beta} + \delta_{\eta}^{\tau} g_{\sigma\lambda}^{\nu\beta} - \delta_{\sigma}^{\tau} g_{\lambda\eta}^{\nu\beta} \tag{26}$$

with

$$g_{\eta\sigma}^{\lambda\alpha} = \delta_{\eta}^{\lambda}\delta_{\sigma}^{\alpha} - \delta_{\sigma}^{\lambda}\delta_{\eta}^{\alpha}. \tag{27}$$

The energy-momentum density is defined by

$$\Xi_{\alpha}^{\beta} = U_{\alpha,\lambda}^{\beta\lambda} \tag{28}$$

where comma denotes ordinary differentiation. The energy is expressed by the surface integral;

$$E = \lim_{r\to\infty} \int_{r=constant} U_0^{0\zeta} \eta_{\zeta} dS \tag{29}$$

where $\eta_{\zeta} (\zeta = 1, 2, 3)$ is the unit three-vector normal to surface element dS. Now, we are interested in to evaluate the total energy distribution. Since the intermediary mathematical exposition are length, we give only the final results. To find the super-potential of Møller, we can firstly evaluate the required the non-vanishing basic vector field Ψ_{μ} and the con-torsion tensor $\xi_{\alpha\beta\lambda}$. After making the some calculations [70, 71], the required non-vanishing components of $\xi_{\alpha\beta\lambda}$ and Ψ_{μ} are obtained as follows

$$\xi_{01}^{0} = -\xi_{11}^{1} = \frac{\Lambda r^4 + 3\chi\mu G(G\mu + r)}{r(\Lambda r^4 - 3\chi(G\mu + r)^2)}, \tag{30}$$

$$\xi_{22}^{1} = -\frac{r}{\sqrt{-\frac{\Lambda r^2}{3} + \chi\left(1 + \frac{G\mu}{r}\right)^2}}, \tag{31}$$

$$\xi^1_{33} = \xi^1_{22} s^2\theta, \tag{32}$$
$$s\theta\xi^2_{21} = \xi^3_{31} = rs\theta, \tag{33}$$
$$\xi^3_{21} = \xi^3_{13} = r^{-1}\sqrt{-\frac{\Lambda r^2}{3} + \chi\left(1+\frac{G\mu}{r}\right)^2}, \tag{34}$$
$$\xi^3_{32} = \xi^3_{23} = \cot\theta, \tag{35}$$
$$\xi^2_{33} = -s\theta c\theta \tag{36}$$

$$\Psi_1 = -\frac{\Lambda r^4 + 3\chi\mu G(G\mu + r)}{r(\Lambda r^4 - 3\chi(G\mu + r)^2)} + \frac{2}{r}\sqrt{-\frac{\Lambda r^2}{3} + \frac{\chi}{r^2}(G\mu + r)^2} \tag{37}$$
$$\Psi_2 = \cot\theta. \tag{38}$$

Using these results with Eq. (24), we obtain the non-vanishing required super-potential of Møller as following

$$U_0^{01} = \frac{r^2 s\theta}{\kappa}\left[-\frac{\Lambda}{3}r - \chi\left(1+\frac{G\mu}{r}\right)\frac{G\mu}{r}\right]. \tag{39}$$

Subsituting above result into the energy-momentum integral, we find the following energy and momentum for the topological black holes, respectively

$$E_M(r) = r\left[-\frac{\Lambda}{3}r^2 - \chi\left(1+\frac{G\mu}{r}\right)\frac{G\mu}{r}\right]. \tag{40}$$

and

$$\Xi^0_1 = \Xi^0_2 = \Xi^0_3 = 0 \tag{41}$$

We can easily see that energy distribution of the topological black holes depends on the charge Q, mass M and cosmological constant Λ since G and μ include Q and M.

4. Special Cases

In this section, we consider equation (22) (or equation (40)) to evaluate the exact solutions for the energy distributions associated with the some special cases of the topological black hole models.

4.1. Topological black holes with $\chi = 1$

For $\chi = 1$, line element (5) becomes [72]

$$ds^2 = \left[-\frac{\Lambda r^2}{3} + \left(1+\frac{G\mu}{r}\right)^2\right]dt^2 - \left[-\frac{\Lambda r^2}{3} + \left(1+\frac{G\mu}{r}\right)^2\right]^{-1}dr^2 - r^2 d\Omega^2 \tag{42}$$

where $0 \leq r < \infty$, $d\Omega^2$ is the metric of S^2, and the scalar field is given by

$$\Phi(r) = \sqrt{\frac{3}{4\pi}}\frac{G^{1/2}}{\frac{r}{M} - G} \tag{43}$$

This solution describes a static and spherically symmetric black hole, provided the cosmological constant Λ is positive.

The energy distribution of this model is found as

$$E_M(r) = r\left[-\frac{\Lambda}{3}r^2 - \left(1+\frac{G\mu}{r}\right)\frac{G\mu}{r}\right]. \tag{44}$$

(i) In the vanishing cosmological constant limit $\Lambda \rightarrow 0$, line element given in equation (42) reduces to the extremal Reissner-Nordstrøm cosmological model [58] defined as

$$ds^2 = \left(1+\frac{G\mu}{r}\right)^2 dt^2 - \left(1+\frac{G\mu}{r}\right)^{-2} dr^2 - r^2 d\Omega^2 \tag{45}$$

In this case, using equations (6) and (7) with $\varsigma = 4\pi$ [58], we find the energy distribution

$$E_M(r) = \left(1-\frac{Q^2}{rM}\right)\frac{Q^2}{M} \tag{46}$$

Energy distribution of the topological black holes depends on their mass M and charge Q. If we take the $r \rightarrow \infty$, energy distribution is calculated as

$$E_M = \frac{Q^2}{M} \tag{47}$$

Furthermore, in the $Q = M$ limit that black hole would be in a state of thermal equilibrium with the background space-time since it has a temperature equal to the de Sitter temperature [75], the energy is reduced to

$$E_M = M \tag{48}$$

The energy depends only on black hole mass M.

(ii) For the neutral (or $Q \rightarrow 0$*) topological black holes*, metric given in equation (42) is reduced to the

$$ds^2 = \left[-\frac{\Lambda r^2}{3} + \left(1+\frac{9\alpha M}{2\pi\Lambda r}\right)^2\right] dt^2 - \left[-\frac{\Lambda r^2}{3} + \left(1+\frac{9\alpha M}{2\pi\Lambda r}\right)^2\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{49}$$

where α is constant and its energy is computed as

$$E_M(r) = r\left[-\frac{\Lambda}{3}r^2 - \left(1+\frac{9\alpha M}{2\pi\Lambda r}\right)\frac{9\alpha M}{2\pi\Lambda r}\right]. \tag{50}$$

The energy depends on the cosmological constant Λ and the black hole mass M.

(iii) For the massless ($M \rightarrow 0$*) and neutral (*$Q \rightarrow 0$*) black holes* , line element (49) becomes

$$ds^2 = \left[1-\frac{\Lambda r^2}{3}\right] dt^2 - \left[1-\frac{\Lambda r^2}{3}\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{51}$$

In this case, we calculate the energy as following

$$E_M(r) = -\frac{\Lambda r^3}{3} \tag{52}$$

We can easily see that in the vanishing cosmological constant limit the energy goes to zero in the above equations. This is an expected result since the line element, under above limits, is reduced to the Minkowski space-time in the spherical coordinates.

4.2. Topological black holes with $\chi = -1$

When the cosmological constant is negative (or $\chi = -1$) , the metric (5) takes the form

$$ds^2 = \left[-\frac{\Lambda r^2}{3} - \left(1 + \frac{G\mu}{r}\right)^2\right] dt^2 - \left[-\frac{\Lambda r^2}{3} - \left(1 + \frac{G\mu}{r}\right)^2\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{53}$$

and describes an asymptotically locally AdS black hole. At the origin $r = 0$ there is a unique curvature singularity. The scalar field is given as

$$\Phi(r) = \sqrt{\frac{1}{2\alpha l^2 r}} \frac{G\mu}{(r + G\mu)} \tag{54}$$

where $\Lambda = -3/l^2$and it has a simple pole at $r = -G\mu$ only if μ is negative and real for $\alpha > 0$. In this case, we can easily find the energy distribution from eq.(22)

$$E_M(r) = r\left[-\frac{\Lambda}{3} r^2 + \left(1 + \frac{G\mu}{r}\right)\frac{G\mu}{r}\right] \tag{55}$$

(i) If we take cosmological constant zero, we obtain the energy distribution, using equations (6) and (7) with $\varsigma = 4\pi$ [58]

$$E_M(r) = \left(\frac{Q^2}{Mr} - 1\right)\frac{Q^2}{M} \tag{56}$$

and for the $r \to \infty$, energy distribution is calculated as

$$E_M = -\frac{Q^2}{M} \tag{57}$$

Furthermore, in the $Q = M$ limit that black hole would be in a state of thermal equilibrium with the background space-time since it has a temperature equal to the de Sitter temperature [75], energy reduced to

$$E_M = -M \tag{58}$$

Energy depends only on black hole mass M. In this model, because of the negative curvature, we have two choices $M > 0$ and $M < 0$ [58]. For the first choice energy is negative and the latter one give us positive energy distribution.

(ii) For the neutral (or $Q \to 0$) topological black holes, metric given in equation (42) is reduced to the

$$ds^2 = \left[-\frac{\Lambda r^2}{3} - \left(1 - \frac{9\alpha M}{2\pi\Lambda r}\right)^2\right] dt^2 - \left[-\frac{\Lambda r^2}{3} - \left(1 - \frac{9\alpha M}{2\pi\Lambda r}\right)^2\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{59}$$

where α is constant and its energy is computed as

$$E_M(r) = r\left[-\frac{\Lambda}{3} r^2 - \left(1 - \frac{9\alpha M}{2\pi\Lambda r}\right)\frac{9\alpha M}{2\pi\Lambda r}\right]. \tag{60}$$

The energy depends on the cosmological constant Λ and black hole mass M.

(iii) For the massless ($M \to 0$) and neutral ($Q \to 0$) black holes , line element (59) is reduced to

$$ds^2 = \left[-1 - \frac{\Lambda r^2}{3}\right] dt^2 - \left[-1 - \frac{\Lambda r^2}{3}\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{61}$$

In this case, we calculate the energy as following

$$E_M(r) = -\frac{\Lambda r^3}{3} \tag{62}$$

We can easily see that in the vanishing cosmological constant line element (61) reduces to the Minkowski space-time in the spherical coordinates. Hence, the energy goes to zero in this limit.

4.3. Topological black holes with $\chi = 0$

Considering $\chi = 0$, the line element (5) becomes

$$ds^2 = \left[-\frac{\Lambda r^2}{3}\right] dt^2 - \left[-\frac{\Lambda r^2}{3}\right]^{-1} dr^2 - r^2 d\Omega^2 \tag{63}$$

Energy distribution, in this model, is found as

$$E_M(r) = -\frac{\Lambda r^3}{3} \tag{64}$$

The energy distribution is actually positive since the cosmological constant has negative sign for the topological black holes with $\chi = 0$ [58]. If the cosmological constant is taken to be zero, then one has

$$E_M(r) = 0. \tag{65}$$

5. Discussion

Since the beginning of relativity, the localization of energy-momentum in general relativity has been debated. Virbhadra [13, 14, 15, 16, 17, 18] underlined that although the energy-momentum prescriptions are not tensorial objects, they do not disturb the principle of general covariance as the equations defining the local conservation laws with these objects are covariant. In another study, Chang, Nester and Chen [73] obtained that there exists a direct relationship between quasilocal and pseudotensor expressions; since every energy-momentum pseudotensor is associated with a legitimate Hamiltonian boundary term. Next, Lessner[59] argued that the Møller energy-momentum complex is a powerful concept of energy and momentum. According to the Cooperstock hypothesis [2], the energy is confined to the region of non-vanishing energy-momentum tensor of matter and all non-gravitational fields.

In this paper, to evaluate the energy distribution(due to matter plus fields) associated with the topological black holes, we investigated the Möller energy-momentum definition in both general relativity and teleparallel gravity. We showed that the energy distribution is the same in both of these different gravitation theories and also found that the energy depends on the mass M and charge Q of the topological black holes and cosmological constant Λ :

$$E_M(r) = r\left[-\frac{\Lambda}{3}r^2 - \chi\left(1 + \frac{G\mu}{r}\right)\frac{G\mu}{r}\right]. \tag{66}$$

Furthermore, we calculated the energy distribution for the some special cases of the topological black holes and found that energy depends on the black holes mass M, charge Q and cosmological constant Λ (equations 44, 46-48, 50, 52, 55-58, 60-61).

Moreover, present paper maintains (a) the importance of the energy-momentum definitions in the evaluation of the energy distribution for a given space-time, (b) the viewpoint of Lessner [74], and c) the Möller energy-momentum definition allows to make calculations in any coordinate system. Finally, the energy obtained is also independent of the teleparallel dimensionless coupling constant, which means that it is valid not only in the teleparallel equivalent of general relativity, but also in any teleparallel model.

References

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and N. Y. Co, N. Y., 1973).

[2] F. I. Cooperstock and R. S. Sarracino, J. of Phys. A **11** (1918), 877.

[3] H. Bondi, Proc. R. Soc. London A **427** (1990), 249

[4] A. Einstein, Sitzungsber. Preus. Akad. Wiss. Berlin (Math. Phys.) **47**, 778 (1915); Addendum-ibid. 799 (1915).

[5] L. D. Landau and E. M. Lifshitz, *The Classical theory of fields* (Pergamon Press, 4th Edition, Oxford, re-printed in 2002).

[6] S. Weinberg, *Gravitation and Cosmology: Principle and Applications of General Theory of Relativity* (John Wiley and Sons, Inc., New York, 1972).

[7] A. Papapetrou, Proc. R. Irish. Acad. **A52**, 11 (1948).

[8] P. G.Bergmann and R. Thomson, Phys. Rev. **89**, 400 (1953).

[9] C. Mϕller, Ann. Phys. (NY) **4**, 347 (1958).

[10] C. Mϕller, Ann. Phys. (NY) **12**,118 (1961).

[11] R.C. Tolman, *Relativity, Thermodinamics and Cosmology* (Oxford Univ. Pres. London, P.227, 1934).

[12] A. Quadir and M. Sharif, Phys. Lett.A,**167**, 331 (1992).

[13] K. S. Virbhadra and N. Rosen, Gen. Rel. Grav. **25**, 429 (1993).

[14] K. S. Virbhadra and A. Chamorro, Pramana-J Phys. **45**, 181 (1995).

[15] K. S. Virbhadra, Phys. Rev. D**41**, 1086 (1990).

[16] K. S. Virbhadra, Phys. Rev. D**42**, 2919 (1990).

[17] K. S. Virbhadra, Phys. Rev. D**60**, 104041 (1999).

[18] K. S. Virbhadra, Pramana-J Phys. **45**, 215 (1995).

[19] E. Vagenas, Int. J. Mod.Phys. A**18**, 5781 (2003).

[20] E. Vagenas, Int. J. Mod. Phys. A**18**, 5949 (2003).

[21] E. Vagenas, Mod. Phys. Lett. A**19**, 213 (2004).

[22] E. Vagenas, Int. J. Mod.Phys. D**14**, 573 (2005).

[23] E. Vagenas,Mod.Phys.Lett. A**21**, 1947 (2006).

[24] S. S. Xulu, hep-th/0308077.

[25] S. S. Xulu, Int. J. Mod. Phys. D**7**, 773 (1998).

[26] S. S. Xulu, Int.J.Mod.Phys. A**15**, 2979 (2000).

[27] I. Radinschi, Fizika B**9**, 43 (2000).

[28] I. Radinschi, Acta Physica Slovaca, **49**, 1 (1999).

[29] I. Radinschi and Th. Grammenos, Int.J.Mod.Phys. A**21**, 2853 (2006).

[30] Th. Grammenos, Mod.Phys.Lett. A**20**, 1741 (2005).

[31] G. Lessner, *Gen. Relativ. Gravit.* **28**, 527 (1996).

[32] F.I. Mikhail, M.I. Wanas, A. Hindawi and E.I. Lashin, Int. J. Theor. Phys. **32**, 1627 (1993).

[33] T. Vargas, Gen. Rel. Gravit. 36, 1255 (2004).
[34] M. Saltı and A. Havare, Int. J Mod. Phys. A**20**, 2169 (2005).
[35] M. Saltı, Astrophys. Space Sci. **299**, 159 (2005).
[36] O.Aydogdu and M. Saltı, Astrophys. Space Sci. **299**, 227 (2005).
[37] M. Saltı, Nuovo Cimento B**120**, 53 (2005).
[38] M. Saltı Mod. Phys. Lett. A**20**, 2175 (2005).
[39] M. Saltı Acta Phys. Slov., **55**, 563 (2005).
[40] O. Aydogdu, Int. J. Mod. Phys. A**21**, 3845 (2006).
[41] O. Aydogdu, Int. J. Mod. Phys. D**15**, 459 (2006).
[42] O. Aydogdu, Fortschritte der Physik **54**, 246 (2006).
[43] O. Aydogdu, M. Saltı and M. Korunur, Acta Phys. Slov. **55**, 537 (2005).
[44] M. Saltı and O. Aydogdu, Int. J. Theor. Phys. **45**, 10 (2006).
[45] M. Saltı and O. Aydogdu, gr-qc/0509061.
[46] M. Saltı and O. Aydogdu, Astrophys. Space Sci. **302**, 61 (2006).
[47] M. Saltı and O. Aydogdu, Prog. Theor. Phys. **115**, 63 (2006).
[48] J. P. S. Lemos, Phys. Lett. B**353**, 46 (1995).
[49] R. G. Cai and Y. Z. Zhang, Phys. Rev. D**54**, 4891 (1996).
[50] L. Vanzo, Phys. Rev. D**56**, 6475 (1997).
[51] D. R. Brill, J. Louko, and P. Peldan, Phys. Rev. D**56**, 3600 (1997).
[52] D. Birmingham, Class. Quant. Grav. **16**, 1197 (1999).
[53] R. G. Cai and K. S. Soh, Phys. Rev. D**59**, 044013 (1999).
[54] J. Crisostomo, R. Troncoso and J. Zanelli, Phys. Rev. D**62**, 084013 (2000).
[55] R. Aros, R. Troncoso and J. Zanelli, Phys. Rev. D**63**, 084015 (2001).
[56] R. G. Cai, Phys. Rev. D**65**, 084014 (2002).
[57] R. G. Cai, Phys.Rev. D**65**, 084014 (2002).
[58] C. Martinez, R. Troncoso and J. P. Staforelli, Phys.Rev. D**74**, 044028 (2006).
[59] H.P. Robertson, Ann. of Math. (Princeton) **33**, 496 (1932).
[60] R. Weitzenböck, *Invariantten theorie* (Gronningen: Noordhoff, 1923).
[61] K. Hayashi and T. Shirafuji, Phys. Rev. D**19**, 3524 (1978).
[62] V.V. de Andrade and J.G. Pereira, Phys. Rev. D**56**, 4689 (1997).
[63] C. Møller, Mat. Fys. Medd. K. Vidensk. Selsk. **39**, 13 (1978).
[64] C. Møller, Mat. Fys. Medd. K. Vidensk. Selsk. **1**, 10 (1961).
[65] D. Saez, Phys. Rev. D**27**, 2839 (1983).
[66] H. Meyer, Gen. Rel. Gravit. **14**, 531 (1982).
[67] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. **64**, 866 (1980).
[68] K. Hayashi and T. Shirafuji, Prog. Theor. Phys. **65**, 525 (1980).
[69] F.W. Hehl, J. Nitsch and P. von der Heyde, *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980).
[70] Wolfram Research, *Mathematica 5.0* (2003).
[71] TCI Software Research, *Scientific Workplace 3.0* (1998).
[72] C. Martinez, R. Troncoso and J. Zanelli, Phys. Rev. D**67**, 024008 (2003).
[73] Chia-Chen Chang, J. M. Nester and Chiang-Mei Chen, Phys. Rev. Lett. **83**, 1897 (1999).
[74] G. Lessner, Gen. Relativ. Gravit. **28**, 527 (1996).
[75] D. Kastor and J. Traschen, Phys.Rev. D**47** (1993), 5370.